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Complex Zeros of an Incomplete Riemann Zeta Function and of the Incomplete Gamma Function

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Abstract. Complex zeros of an incomplete Riemann zeta function and of the incomplete gamma function are calculated as functions of the upper limit λ of the definition integrals. It becomes apparent that not all, but only some, of the zero trajectories of the incomplete Riemann zeta function join a zero of the Riemann zeta function $\zeta(s)$ for $\lambda \to \infty$. The remaining trajectories at least in the region considered, approach the zero trajectories of the incomplete gamma function.

1. Introduction. Let $s = \sigma + it$ be a complex variable. The gamma function can then be defined for $\operatorname{Re} s = \sigma > 0$ by the infinite integral

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x}\, dx. \tag{1}$$

By replacing the upper limit of the integral by a parameter $\lambda > 0$, one obtains the so-called incomplete gamma function [1]

$$\gamma(s, \lambda) = P(s, \lambda)\Gamma(s) = \int_0^\lambda x^{s-1} e^{-x}\, dx \tag{2}$$

or

$$P(s, \lambda) = \frac{1}{\Gamma(s)} \int_0^\lambda x^{s-1} e^{-x}\, dx. \tag{3}$$

This function plays an important role in several fields.

It seems to be much less well known that a similar procedure can be applied to the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}. \tag{4}$$

This function can be defined for $\sigma > 1$ by the infinite integral

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx. \tag{5}$$

Replacing the upper limit by $\lambda > 0$ gives

$$A(s, \lambda) = \frac{1}{\Gamma(s)} \int_0^\lambda \frac{x^{s-1}}{e^x - 1}\, dx.^* \tag{6}$$

We shall call this function the incomplete Riemann zeta function.

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* For integers $s = p$ ($p \geqq 2$), $A(p, \lambda)$ is related to the Debye functions [1].

Properties of $A(s, \lambda)$ have been investigated by Putschbach [2] in a manuscript, which, as far as the author knows, remains unpublished.

It is known that the imaginary parts t_m of the first nontrivial zeros of the Riemann zeta function $\zeta(s)$ on the line $\sigma = \frac{1}{2}$ are given by $t_1 = 14.13473$, $t_2 = 21.02204$, $t_3 = 25.01086, \cdots$ [3]. Since

$$\lim_{\lambda\to\infty} A(s, \lambda) = \zeta(s),$$

it would be interesting to know how the zeros of $A(s, \lambda)$, if there are any, approach the zeros of $\zeta(s)$, if in fact they do.

It is the aim of this paper to show, by numerical calculation, the behaviour of some of the solutions $\bar{s}(\lambda)$ of $A(\bar{s}(\lambda), \lambda) = 0$ in the s-plane. In particular, it will be seen that not all of these functions $\bar{s}(\lambda)$ approach a nontrivial zero of $\zeta(s)$, but only some of them. The remaining curves, at least in the region considered, approach the zero trajectories $\bar{\bar{s}}(\lambda)$ of the incomplete gamma function $P(s, \lambda)$ as $\lambda \to \infty$.

2. **Other Formulae for $A(s, \lambda)$ and $P(s, \lambda)$.** The definitions (3) and (6) are valid only for $\sigma > 0$ and $\sigma > 1$, respectively. Using the series expansion for the exponential function in Eq. (3), one finds for $\sigma > 0$

$$\text{(7)} \qquad \begin{aligned} P(s, \lambda) &= \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\lambda^{s+n}}{s+n} \\ &= \frac{\lambda^s}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \lambda^n \frac{1}{s+n}. \end{aligned}$$

By analytic continuation, this formula now defines $P(s, \lambda)$ for $|s| < \infty$ and $0 < \lambda < \infty$, with removable singularities at $s = k$ $(k = 0, -1, -2, \cdots)$, where $P(k, \lambda) = 1$. For the incomplete zeta function (6), this procedure becomes more complicated. Taking the series

$$\text{(8)} \qquad \frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n,$$

where the B_n are the Bernoulli numbers, we see that this series converges for $|x| < 2\pi$ only. We therefore split the integration interval into two subintervals: 0 to λ' and λ' to λ, where $0 < \lambda' < 2\pi$. We substitute the series (8) into the first of these two integrals and obtain

$$\text{(9)} \qquad A(s, \lambda) = \frac{1}{\Gamma(s)} \left\{ \sum_{n=0}^{\infty} \frac{B_n}{n!} \frac{\lambda'^{s+n-1}}{s+n-1} + \int_{\lambda'}^{\lambda} \frac{x^{s-1}}{e^x - 1} dx \right\}.$$

For $\lambda' = 1$, this formula gives

$$\text{(10)} \qquad A(s, \lambda) = \frac{1}{\Gamma(s)} \left\{ \sum_{n=0}^{\infty} \frac{B_n}{n!} \frac{1}{s+n-1} + \int_{1}^{\lambda} \frac{x^{s-1}}{e^x - 1} dx \right\}$$

and, for $\lambda \to \infty$,

$$\text{(11)} \qquad \zeta(s) = \frac{1}{\Gamma(s)} \left\{ \sum_{n=0}^{\infty} \frac{B_n}{n!} \frac{1}{s+n-1} + \int_{1}^{\infty} \frac{x^{s-1}}{e^x - 1} dx \right\}.$$

This last formula for $\zeta(s)$ corresponds to the decomposition of Prym for $\Gamma(s)$ [4].

We see here that Eqs. (9), (10) and (11) are valid also in the half plane $\sigma \leqq 1$, except at the integer points $s = 1, 0, -1, -2, \cdots$.

For integers $s = 1 - k$ ($k \geqq 1$) one finds from Eq. (10) using the limit

$$\lim_{s \to 1-k} (s + k - 1)\Gamma(s) = \frac{(-1)^{k-1}}{(k-1)!}, \tag{12}$$

and the fact that the integrals in Eqs. (10) and (11) are bounded, the result

$$A(-k, \lambda) = \zeta(-k) = (-1)^k \frac{B_{k+1}}{k+1} \tag{13}$$

for all λ and $k = 0, 1, 2, \cdots$. In particular, we have

$$\begin{aligned} A(0, \lambda) &= -\tfrac{1}{2}, \\ A(-2k, \lambda) &= 0 \qquad (k \geqq 1), \\ A(-2k+1, \lambda) &= -\frac{B_{2k}}{2k} \qquad (k \geqq 1), \end{aligned} \tag{14}$$

and, in addition, $A(1, \lambda) = \zeta(1) = \infty$.

With the help of the expansion

$$\frac{1}{e^x - 1} = \sum_{n=1}^{\infty} e^{-nx} \qquad (x > 0) \tag{15}$$

we obtain from Eq. (6)

$$A(s, \lambda) = \frac{1}{\Gamma(s)} \sum_{n=1}^{\infty} \int_0^{\lambda} x^{s-1} e^{-nx}\, dx \qquad (\sigma > 1). \tag{16}$$

Making the substitution $x' = nx$ and using Eq. (3) we have

$$A(s, \lambda) = \sum_{n=1}^{\infty} n^{-s} P(s, n\lambda). \tag{17}$$

This relation was also found by Putschbach [2].

For the asymptotic behaviour of $P(s, \lambda)$ we find in [4] for Re $s \to \infty$

$$P(s, \lambda) = e^s \lambda^s s^{-s-1/2} e^{-\lambda} \frac{1}{(2\pi)^{1/2}} \{1 + O(1/s)\} \tag{18}$$

so that

$$A(s, \lambda) = \frac{1}{(2\pi)^{1/2}} e^s \lambda^s s^{-s-1/2} \sum_{n=1}^{\infty} e^{-n\lambda} \{1 + O(1/s)\}. \tag{19}$$

For large, but fixed, λ it follows for Re $s \to \infty$ that

$$A(s, \lambda) \approx P(s, \lambda) \approx \frac{1}{(2\pi)^{1/2}} e^{s-\lambda} \lambda^s s^{-s-1/2}. \tag{20}$$

3. Nontrivial Zeros of A(s, λ). Putschbach [2] observed that for real negative s the function $A(s, \lambda)$ oscillates about the function $\zeta(s)$, and both functions have the points defined by Eq. (13) in common. Furthermore, he found that $A(s, \lambda)$ has two

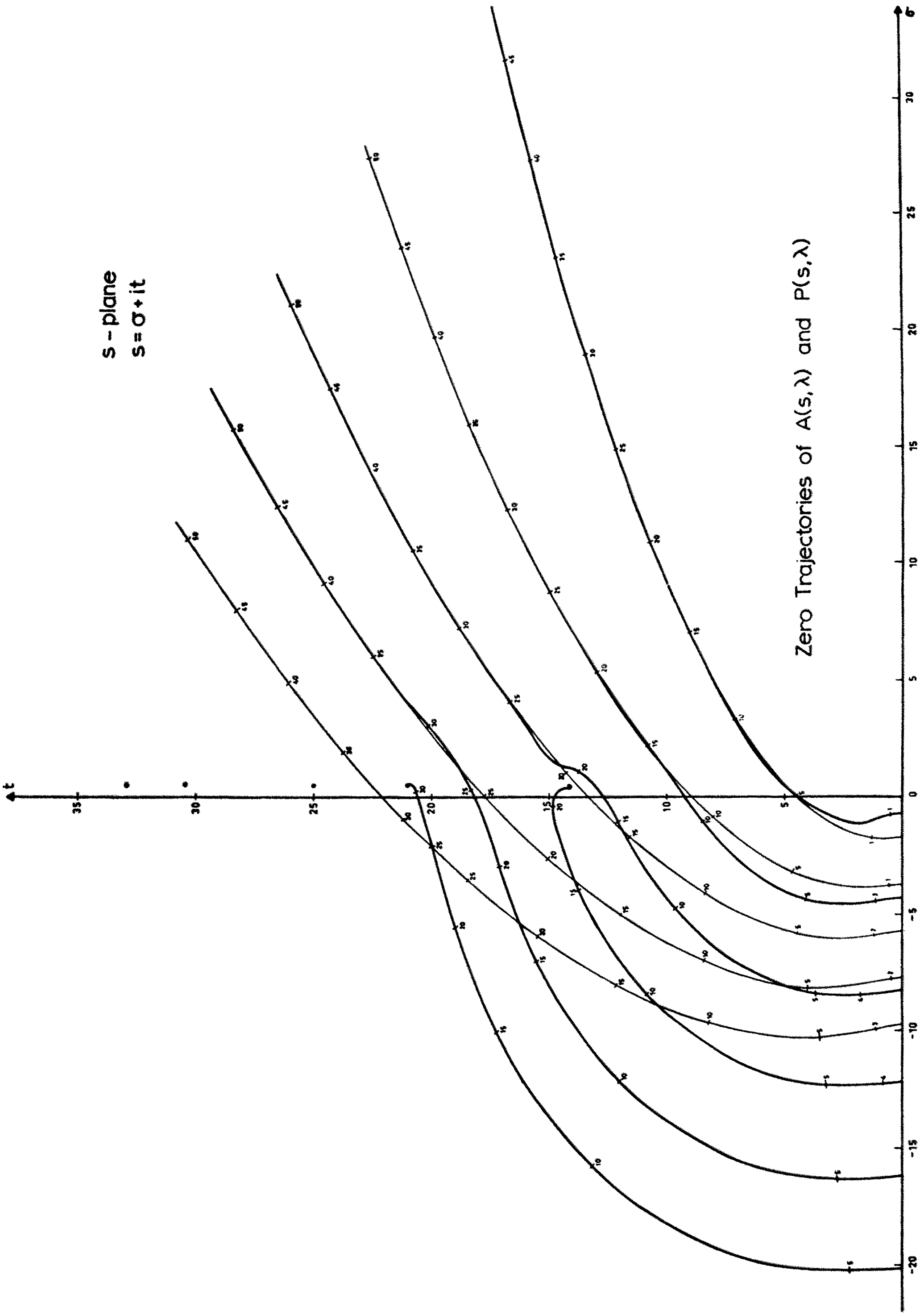


FIG. 1

zeros in the interval $-1 < s < 0$, provided $\lambda \lesssim 0.6$. As λ increases, these two zeros eventually coincide in a double zero $\bar{s}^*$, for $\lambda = \lambda^*$. For $\lambda > \lambda^*$, a pair of complex conjugate zeros moves into the s-plane. The abscissa $\bar{s}^*$ marks the beginning of a trajectory $\bar{s}_1(\lambda)$ in the half plane $t > 0$, defined by $A(\bar{s}_1(\lambda), \lambda) = 0$. Putschbach found also that a second trajectory $\bar{s}_2(\lambda)$ starts in the interval $-5 < s < -4$. By a numerical investigation, one finds that the starting points on the negative real axis for the first six trajectories $\bar{s}_m$ are given in Table 1.

TABLE 1

m	λ_m^*	$\bar{s}_m^* = \bar{s}_m(\lambda_m^*)$
1	0.65687	−0.68126
2	2.53514	−4.29987
3	3.35107	−8.21375
4	3.84904	−12.17003
5	4.19172	−16.14216
6	4.44408	−20.12259

The behaviour of $A(s, \lambda)$ near $\bar{s}_1^*$ and $\bar{s}_2^*$ is shown in Figs. M1 and M2. Figure M1 is taken from Putschbach [2]. It seems that $\bar{s}_m^* = 4 - 4m - \epsilon(m)$, where $\epsilon(m) = O(1)$ for $m \to \infty$; no attempt was made to prove this relation.

The first trajectory $\bar{s}_1(\lambda)$ has been approximately calculated by Putschbach [2] up to $\lambda = 5$. He indicated that each of the curves $\bar{s}_m(\lambda)$ would probably approach a nontrivial zero of $\zeta(s)$ for $\lambda \to \infty$. The author [5] extended the calculation of $\bar{s}_1(\lambda)$ up to $\lambda = 9$. The behaviour of this trajectory in the interval $5 \leqq \lambda \leqq 9$ showed that a direct approach to the first zero of $\zeta(s)$ seemed unlikely.

A systematic investigation of the first six trajectories $\bar{s}_m(\lambda)$ up to $\lambda = 50$ gives the result shown in Fig. 1. The first, second, third and fifth curves, $\bar{s}_1(\lambda)$, $\bar{s}_2(\lambda)$, $\bar{s}_3(\lambda)$ and $\bar{s}_5(\lambda)$, do not approach a zero of $\zeta(s)$, but continue into the half plane $\sigma > 1$. The fourth and the sixth curve $\bar{s}_4(\lambda)$ and $\bar{s}_6(\lambda)$, however, reach for increasing λ the zeros $s_1 = \frac{1}{2} + 14.13473i$ and $s_2 = \frac{1}{2} + 21.02204i$ of $\zeta(s)$, respectively. Therefore, there is no direct correspondence between the zeros $\bar{s}_m(\lambda)$ of $A(s, \lambda)$ for $\lambda \to \infty$ and the zeros s_m of $\zeta(s)$ on the line $\sigma = \frac{1}{2}$.

It is interesting to observe from Fig. 1 that the picture of the $\bar{s}_m(\lambda)$ curves which shows a uniform behaviour in the half plane $\sigma \lesssim -2$, becomes considerably distorted in the region around $\sigma = \frac{1}{2}$. In the half plane $\sigma \gtrsim 5$, the curves are again smooth.

The curves $\bar{s}_4(\lambda)$ and $\bar{s}_6(\lambda)$ approach the corresponding zeros s_1 and s_2 of $\zeta(s)$ in different ways, as can be seen from Fig. 1 and, in detail, from Figs. M3 and M4.

4. **Zeros of $P(s, \lambda)$.** An examination of the graph of the function

$$\gamma^*(s, \lambda) = \lambda^{-s} P(s, \lambda) \tag{21}$$

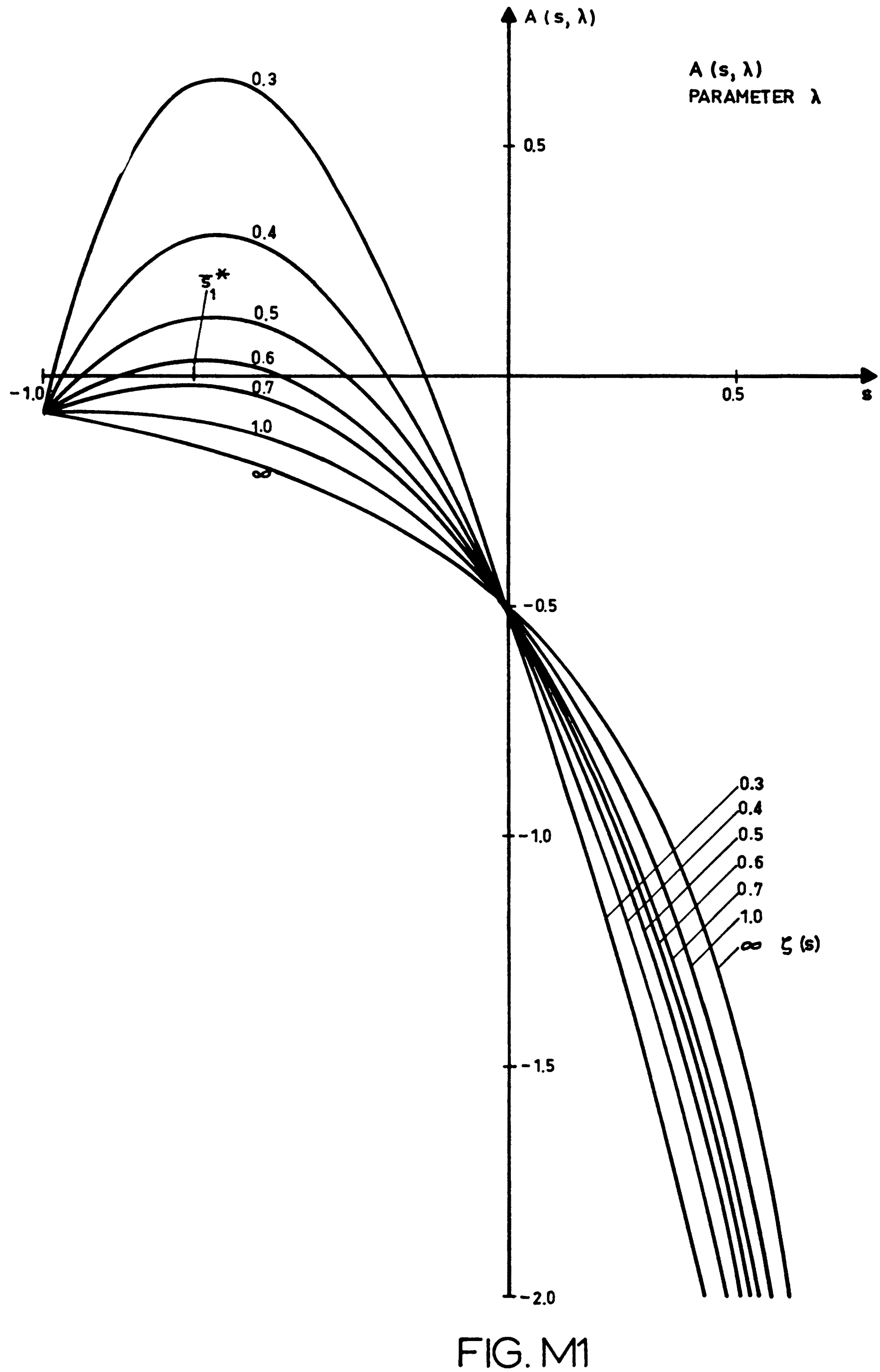

A (s, λ)
A (s, λ)
PARAMETER λ
0.3
0.4
0.5
0.6
0.7
1.0
∞
ξ*
1
0.5
-0.5
-1.0
-1.5
-2.0
-1.0
0.5
s
0.3
0.4
0.5
0.6
0.7
1.0
∞
ζ (s)

FIG. M1

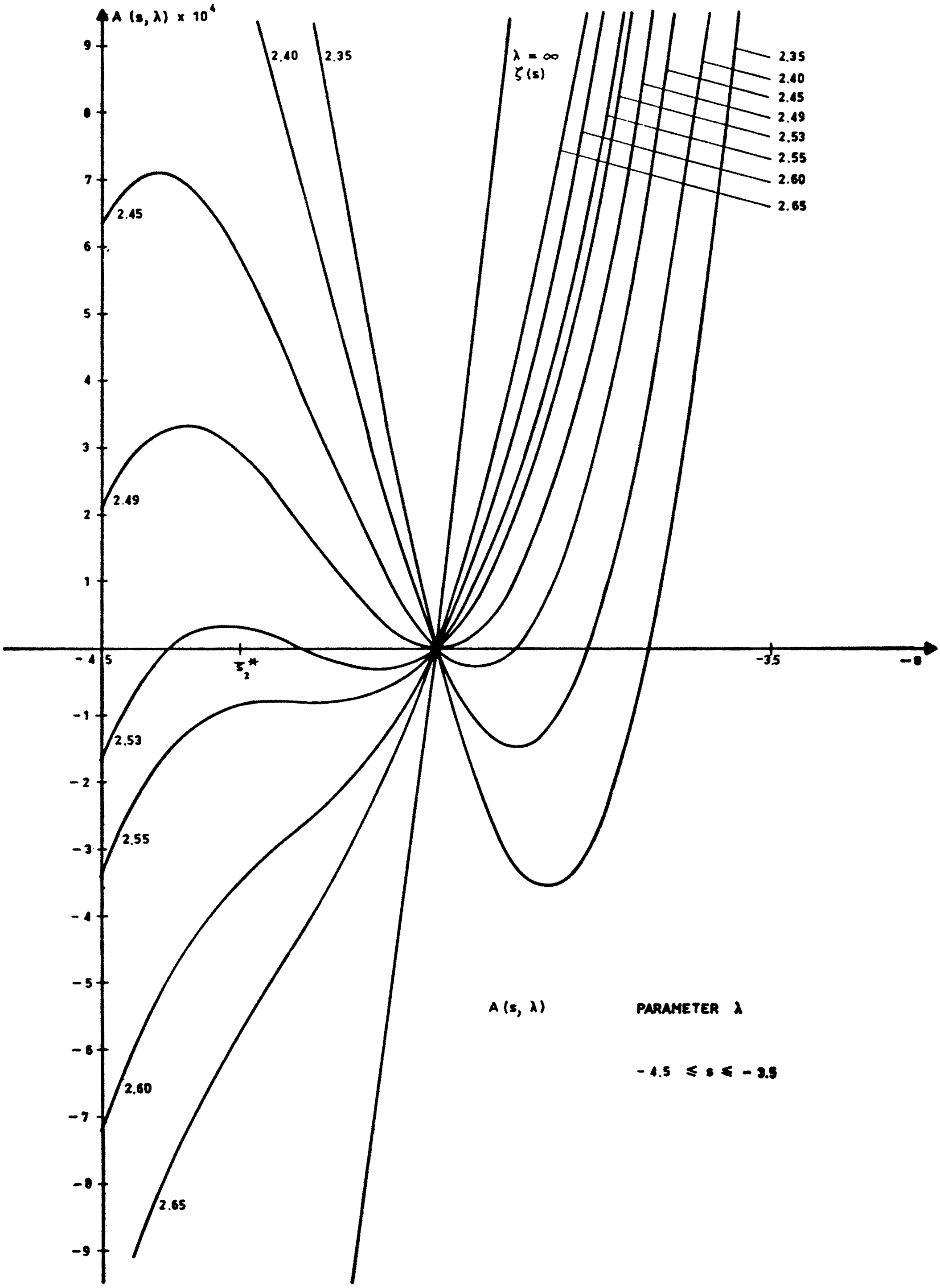
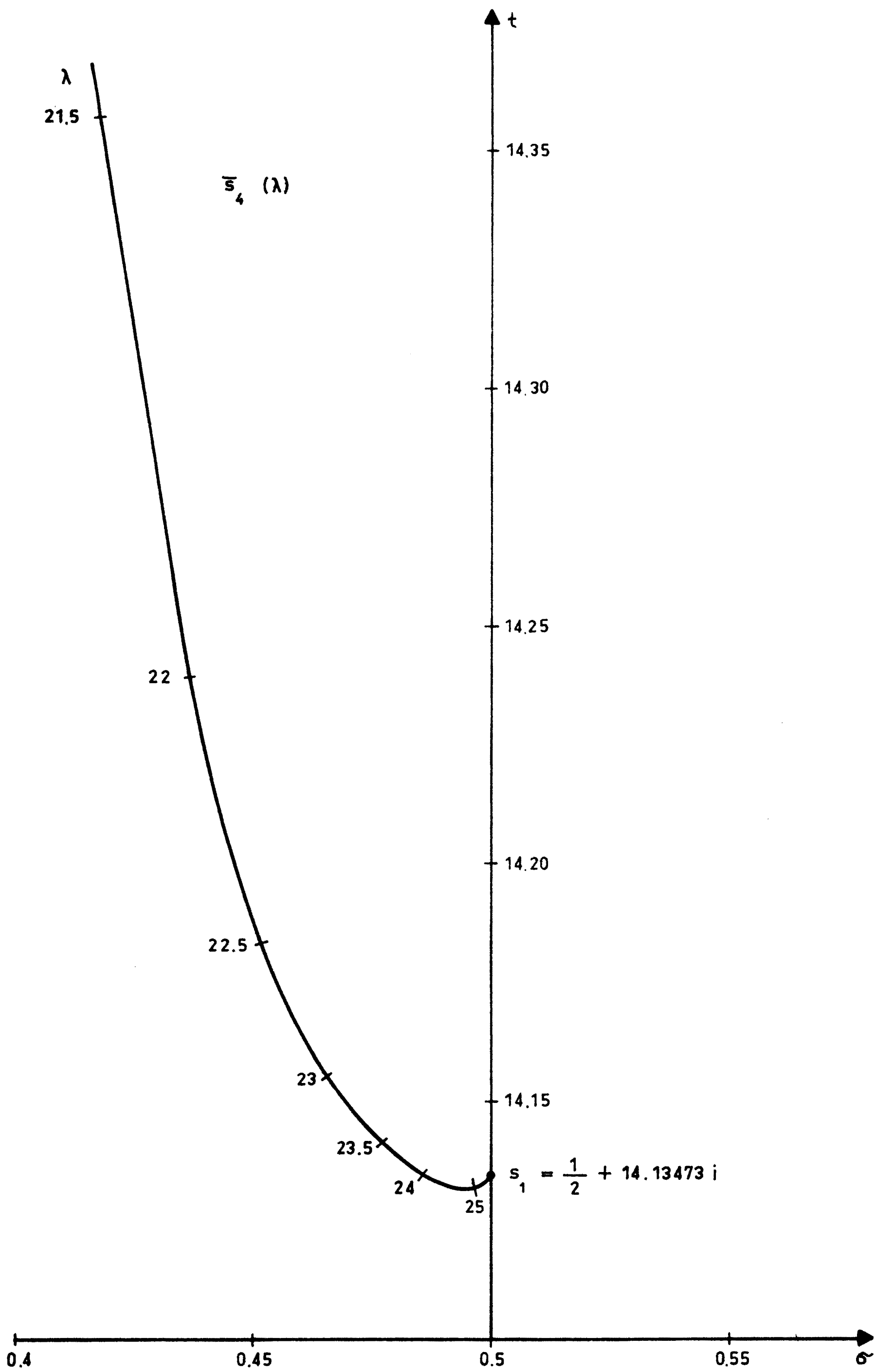
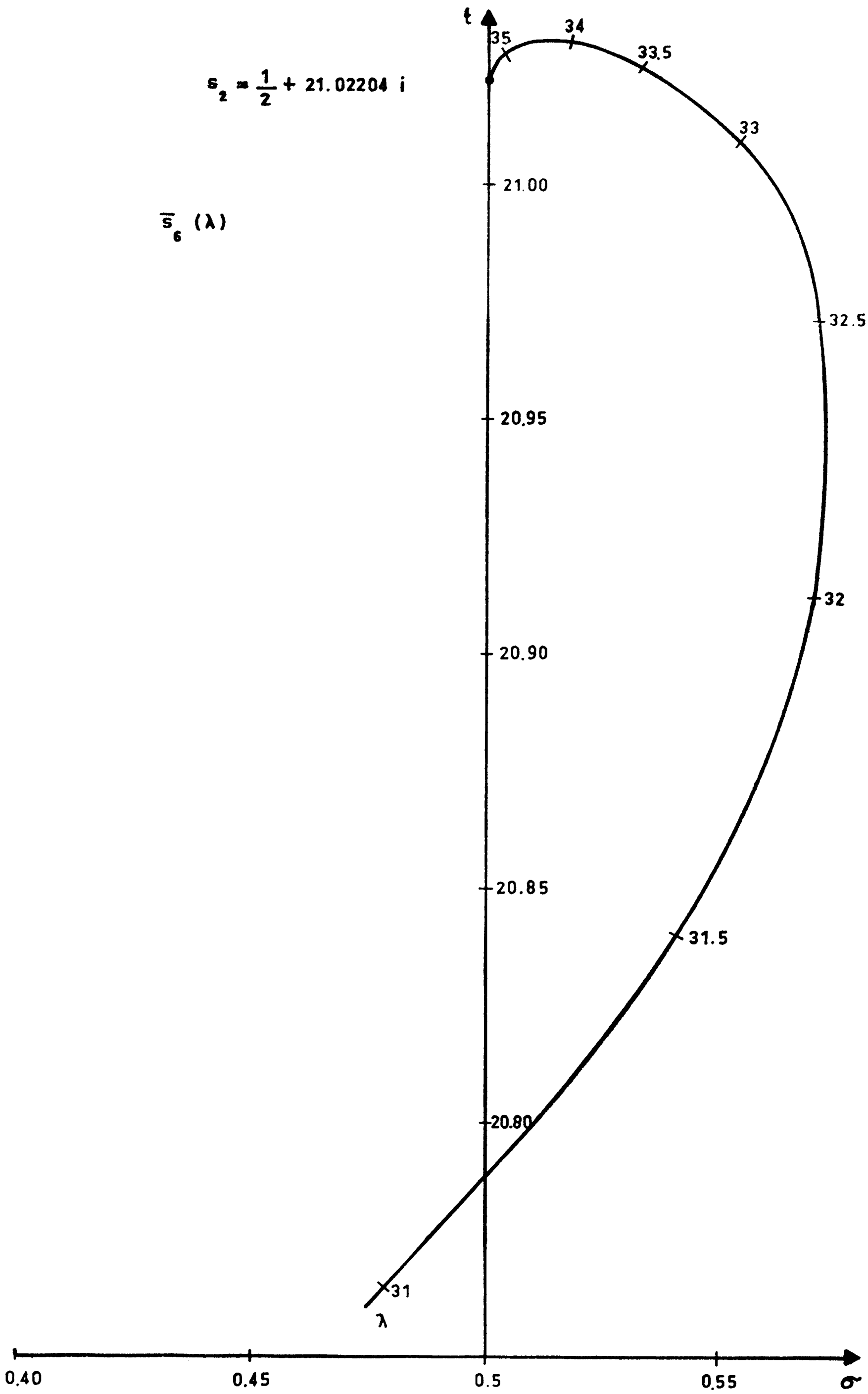

A (s, λ) × 10^4
9
8
7
6
5
4
3
2
1
-1
-2
-3
-4
-5
-6
-7
-8
-9
-4.5
-3.5
-s
2.40
2.35
λ = ∞
ζ (s)
2.35
2.40
2.45
2.49
2.53
2.55
2.60
2.65
2.45
2.49
2.53
2.55
2.60
2.65
A (s, λ)
PARAMETER λ
- 4.5 ≤ s ≤ - 3.5

FIG. M2



t
λ
21.5
$\bar{s}_4$ (λ)
14.35
14.30
14.25
22
14.20
22.5
23
14.15
23.5
24
25
$s_1 = \frac{1}{2} + 14.13473\ i$
0.4
0.45
0.5
0.55
σ

FIG. M3



t
35
34
33.5
33
$s_2 = \frac{1}{2} + 21.02204\ i$
21.00
$\bar{s}_6\ (\lambda)$
32.5
20.95
32
20.90
20.85
31.5
20.80
31
λ
0.40
0.45
0.5
0.55
σ

FIG. M4

which was originally published by Tricomi [6], and which was reproduced in [1], shows that $\gamma^*(s, \lambda)$—and hence $P(s, \lambda)$—has double zeros in the intervals $-2 < s < -1$ and $-4 < s < -3$, in both cases for $0 < \lambda < 1$. Furthermore, one can see from the results obtained by Tricomi [6], [7] that $P(s, \lambda)$ has a positive zero $\lambda_0(s)$ for each value of s satisfying

$$-2m < s < 1 - 2m \qquad (m = 1, 2, 3, \cdots).$$

This function $\lambda_0(s)$ has a maximum value λ_m^* at a certain value $s = \tilde{s}_m^*$; this means $P(s, \lambda)$ has a double zero at $\tilde{s}_m^*$ for $\lambda = \lambda_m^*$. As in the case of the incomplete zeta function, these double zeros are starting points for trajectories $\tilde{s}_m(\lambda)$ with $P(\tilde{s}_m(\lambda), \lambda) = 0$ in the s-plane. A numerical investigation then gives the starting points for the first five trajectories $\tilde{s}_m(\lambda)$ as shown in Table 2.

TABLE 2

m	λ_m^*	$\tilde{s}_m^* = \tilde{s}_m(\lambda_m^*)$
1	0.30809	−1.64425
2	0.77997	−3.63887
3	1.28634	−5.63573
4	1.80754	−7.63372
5	2.33692	−9.63230

Tricomi [6], [7] has shown that the following asymptotic expression for the positive zero $\lambda_0(s)$ holds for $s \to -\infty$ in the above intervals

$$\lambda_0(s) \sim -\tau s - \frac{\tau}{1 + \tau} \log \left[\frac{1 + \tau(-\pi s/2)^{1/2}}{\sin \pi s} \right]. \tag{22}$$

$\tau = 0.27846$ is the real root of the equation

$$1 + x + \log x = 0.$$

A first approximation to the maximum abscissa $\tilde{s}_m^*$ can be found by differentiation of Eq. (22). This gives the following equation for $s = \tilde{s}_m^*$

$$\frac{\tau\pi/2}{(-2\pi s)^{1/2} - \tau\pi s} + \pi \cot \pi s = 1 + \tau. \tag{23}$$

For large $| s |$ one can neglect the first term and obtain for the remaining equation the solution

$$\begin{aligned} \tilde{s}_m^* &\sim 1 - 2m - \frac{1}{2} - \frac{1}{\pi} \arctan \frac{1 + \tau}{\pi} \\ &\sim 1 - 2m - 0.62302. \end{aligned} \tag{24}$$

A comparison with Table 2 shows that for $m = 5$ the error is less than 0.1%. Introducing Eq. (24) into Eq. (22) gives for $m \to \infty$

$$\lambda_m^* \sim 2\tau m - \frac{1}{2} \frac{\tau}{1 + \tau} \log m + 0.03213. \tag{25}$$

The trajectories $\tilde{s}_m(\lambda)$ have been calculated for the given values of m up to $\lambda = 50$. They are shown in Fig. 1. These curves behave much more uniformly than those of the incomplete zeta function. In addition, they join, at least in the region investigated, those zero trajectories $\bar{s}_m(\lambda)$ of $A(s, \lambda)$ which do not end in a zero of $\zeta(s)$. This means there is a correspondence between $\bar{s}_1(\lambda)$ and $\tilde{s}_1(\lambda)$, $\bar{s}_2(\lambda)$ and $\tilde{s}_2(\lambda)$, $\bar{s}_3(\lambda)$ and $\tilde{s}_3(\lambda)$, and between $\bar{s}_5(\lambda)$ and $\tilde{s}_4(\lambda)$.

5. The Numerical Calculation of the Zero Trajectories. The expression (7) for $P(s, \lambda)$ and the expression (10) for $A(s, \lambda)$ were used for the evaluation of $\tilde{s}_m(\lambda)$ and of $\bar{s}_m(\lambda)$, respectively. Most of the calculations were carried out in double-precision arithmetic on a CDC 6600 computer, corresponding to about 28 decimal digits. For $\tilde{s}_4(\lambda)$ and $\tilde{s}_5(\lambda)$ with $\lambda > 25$, it became apparent that the series (7) is no longer suitable, because of the fact that the value of $P(s, \lambda)$ lies in the round-off region. In this case, the relation

$$P(s, \lambda) = \frac{1}{\Gamma(s)} \left\{ \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{s+n} + \int_1^{\lambda} x^{s-1} e^{-x} \, dx \right\} \tag{26}$$

was used.

The integrals in Eqs. (10) and (26) were calculated with an adaptive double-precision Gaussian integration routine [8]. The summation in Eq. (7) was carried out until the modulus of the coefficient

$$c_n = \frac{(-1)^n}{n!} \lambda^n$$

became for fixed λ less than 10^{-30}. The upper limit for n in Eq. (10) was taken to be 60, which is a safe value. This can be seen from the asymptotic behaviour of the Bernoulli numbers and from $B_{60}/60! \approx 0.26 \times 10^{-47}$. The Bernoulli numbers were taken from [1]. The sum in Eq. (26) was calculated as far as $n = 40$. In fact, the time for summing the series in Eqs. (10) or (26) is quite negligible in comparison to the time required for the computation of the integrals.

For the determination of the zeros, a library program written by Gérard [9] for solving a system of nonlinear equations by Newton's method was applied to the two equations

$$\operatorname{Re} A(s, \lambda) = 0, \qquad \operatorname{Im} A(s, \lambda) = 0,$$

for finding $\bar{s}_m(\lambda)$, and to

$$\operatorname{Re} P(s, \lambda) = 0, \qquad \operatorname{Im} P(s, \lambda) = 0,$$

for finding $\tilde{s}_m(\lambda)$. In both cases the unknowns were σ and t. Since λ^s and $1/\Gamma(s)$ have no zeros in the region of the complex plane considered, only the moduli of these functions were taken into account (as convenient scaling factors).

In order to find the starting points $\bar{s}_m^*$ and $\tilde{s}_m^*$, the systems

$$A(s, \lambda) = 0, \qquad \frac{d}{ds} A(s, \lambda) = 0$$

and

$$P(s, \lambda) = 0, \qquad \frac{d}{ds} P(s, \lambda) = 0$$

were solved for the unknowns $s = \sigma$ and λ, using the same program.

TABLE M1

Zero trajectories for the incomplete zeta function $A(s, \lambda)$

$\bar{s}_1(\lambda)$			$\bar{s}_2(\lambda)$		
λ	σ	t	λ	σ	t
1	−0.78560	0.48873			
2	−1.02885	1.36806			
3	−0.99776	2.35217	3	−4.46979	1.11093
4	−0.68749	3.31309	4	−4.59882	2.63643
5	−0.18279	4.17907	5	−4.35230	4.03199
6	0.44072	4.93346	6	−3.85580	5.24344
7	1.12927	5.58612	7	−3.21742	6.26612
8	1.84958	6.15643	8	−2.51077	7.12240
9	2.58339	6.66439	9	−1.78191	7.84322
10	3.32208	7.12696	10	−1.05779	8.45942
11	4.06267	7.55706	11	−0.35287	8.99846
12	4.80511	7.96383	12	0.32647	9.48381
13	5.55062	8.35329	13	0.97926	9.93510
14	6.30061	8.72911	14	1.60862	10.36853
15	7.05622	9.09343	15	2.22093	10.79675
16	7.81810	9.44742	16	2.82516	11.22829
17	8.58644	9.79182	17	3.43155	11.66670
18	9.36111	10.12713	18	4.04914	12.11019
19	10.14181	10.45382	19	4.68307	12.55328
20	10.92813	10.77237	20	5.33384	12.99015
21	11.71967	11.08325	21	5.99901	13.41705
22	12.51604	11.38692	22	6.67543	13.83272
23	13.31692	11.68382	23	7.36060	14.23743
24	14.12199	11.97436	24	8.05285	14.63212
25	14.93099	12.25893	25	8.75119	15.01776
30	19.02744	13.60343	30	12.31940	16.83590
35	23.19492	14.84000	35	15.99564	18.50829
40	27.41892	15.99107	40	19.75974	20.06558
45	31.68930	17.07221	45	23.59633	21.52848
50	35.99862	18.09478	50	27.49400	22.91220

The search for the zeros along the trajectories $\bar{s}_m(\lambda)$ and $\bar{\bar{s}}_m(\lambda)$ was made in the following way. From an approximate value s' for a given λ the program of Gérard calculated the true value $\bar{s}_m$ or $\bar{\bar{s}}_m$ by iteration to five decimals. The approximation s' was found in several ways: using previous calculations [2], [5], using the initial values given in Tables 1 and 2, or, in some cases (mainly for $m \leqq 3$ and $\lambda > 25$) by an extrapolation from the trajectories already calculated. Once the true solution $\bar{s}_m$ or $\bar{\bar{s}}_m$ had been obtained, λ was replaced by $\lambda + \Delta\lambda$ and the solution $s(\lambda)$ was used

TABLE M1 (*cont.*)

$\bar{s}_3(\lambda)$			$\bar{s}_4(\lambda)$		
λ	σ	t	λ	σ	t
4	−8.42035	1.74239	4	−12.22581	0.76351
5	−8.37020	3.65007	5	−12.35540	3.22021
6	−7.93471	5.31811	6	−11.97894	5.34229
7	−7.26889	6.72371	7	−11.28007	7.13550
8	−6.48265	7.88720	8	−10.40516	8.61869
9	−5.64675	8.84491	9	−9.44945	9.83465
10	−4.80426	9.63483	10	−8.47122	10.83040
11	−3.98037	10.29071	11	−7.50427	11.64860
12	−3.18957	10.84054	12	−6.56699	12.32494
13	−2.44042	11.30675	13	−5.66830	12.88786
14	−1.73880	11.70746	14	−4.81148	13.35939
15	−1.09012	12.05809	15	−3.99638	13.75598
16	−0.50059	12.37351	16	−3.22067	14.08917
17	0.02257	12.67069	17	−2.48032	14.36568
18	0.47204	12.97140	18	−1.76937	14.58645
19	0.84086	13.30522	19	−1.07841	14.74293
20	1.11887	13.72180	20	−0.39060	14.80223
21	1.31817	14.39451	21	0.29134	14.60119
22	1.97996	15.22736	22	0.43681	14.23920
23	2.69664	15.77953	23	0.46538	14.15516
24	3.37412	16.26119	24	0.48555	14.13470
25	4.03031	16.71535	25	0.49548	14.13199
30	7.25345	18.85758	30	0.50003	14.13472
35	10.57523	20.86303	35	0.50000	14.13473
40	14.00914	22.73530	40	0.50000	14.13473
45	15.53357	24.49459	45	0.50000	14.13473
50	21.13411	26.15909	50	0.50000	14.13473

as an initial approximation s' to $s(\lambda + \Delta\lambda)$. The increment $\Delta\lambda$ was chosen empirically to be 0.25. In a final run, the functions $A(s, \lambda)$ and $P(s, \lambda)$ were calculated for the four arguments $s = s_m(\lambda) \pm 10^{-5}$ and $s = s_m(\lambda) \pm 10^{-5}i$. Simultaneous sign changes in the real and the imaginary parts of the functions were found in all cases.

Table M1 gives five-digit values of the zeros of $A(s, \lambda)$ for the first six trajectories $\bar{s}_m(\lambda)$. The values for the zeros of $P(s, \lambda)$ for the first five trajectories $\bar{\bar{s}}_m(\lambda)$ are given in Table M2.

The general behaviour of the other trajectories $\bar{s}_m(\lambda)$ for $m > 6$ and $\bar{\bar{s}}_m(\lambda)$ for $m > 5$ seems to be unknown. It is probable that $\bar{s}_7(\lambda)$ joins $\bar{\bar{s}}_5(\lambda)$ and that $\bar{s}_8(\lambda)$ ends in the third zero of $\zeta(s)$, that is, at $s_3 = \frac{1}{2} + 25.01086i$. Of course, it would be of

Table M1 (*cont.*)

$\bar{s}_5(\lambda)$			$\bar{s}_6(\lambda)$		
λ	σ	t	λ	σ	t
5	−16.32207	2.76585	5	−20.27733	2.29409
6	−16.00875	5.34149	6	−20.03007	5.32337
7	−15.28018	7.52358	7	−19.27452	7.89489
8	−14.31946	9.32943	8	−18.23036	10.02442
9	−13.24691	10.80833	9	−17.04310	11.76753
10	−12.13694	12.01640	10	−15.80341	13.18927
11	−11.03325	13.00557	11	−14.56512	14.35053
12	−9.96006	13.81983	12	−13.35845	15.30326
13	−8.92959	14.49492	13	−12.19913	16.08984
14	−7.94686	15.05923	14	−11.09411	16.74406
15	−7.01266	15.53517	15	−10.04522	17.29263
16	−6.12530	15.94041	16	−9.05141	17.75653
17	−5.28173	16.28904	17	−8.11007	18.15231
18	−4.47809	16.59257	18	−7.21791	18.49308
19	−3.71009	16.86083	19	−6.37143	18.78927
20	−2.97327	17.10295	20	−5.56728	19.04931
21	−2.26341	17.32832	21	−4.80245	19.28000
22	−1.57738	17.54750	22	−4.07449	19.48688
23	−0.91446	17.77246	23	−3.38168	19.67441
24	−0.27737	18.01521	24	−2.72327	19.84601
25	0.32901	18.28536	25	−2.09984	20.00399
26	0.90173	18.58923	26	−1.51390	20.14945
27	1.44362	18.93147	27	−0.97106	20.28241
28	1.96507	19.31596	28	−0.48192	20.40298
29	2.48362	19.74262	29	−0.06373	20.51502
30	3.01833	20.20151	30	0.26240	20.63091
35	5.98277	22.51928	31	0.47862	20.76608
40	9.15331	24.66536	32	0.57021	20.91242
45	12.42733	26.67402	33	0.55305	21.00902
50	15.78768	28.57536	34	0.51789	21.03077
			35	0.50341	21.02830
			36	0.49979	21.02461
			37	0.49944	21.02279
			38	0.49968	21.02217
			39	0.49988	21.02202
			40	0.49996	21.02201

TABLE M2

Zero trajectories for the incomplete gamma function $P(s, \lambda)$

$\mathfrak{s}_1(\lambda)$			$\mathfrak{s}_2(\lambda)$		
λ	σ	t	λ	σ	t
1	−1.72630	1.23809	1	−3.72647	0.54067
2	−1.45710	2.29282	2	−3.88891	1.83256
3	−1.01952	3.13777	3	−3.77348	2.89397
4	−0.49804	3.86483	4	−3.51978	3.82369
5	0.07663	4.51288	5	−3.18074	4.66264
6	0.68922	5.10302	6	−2.78218	5.43355
7	1.33072	5.64833	7	−2.33889	6.15075
8	1.99532	6.15758	8	−1.86027	6.82406
9	2.67894	6.63702	9	−1.35275	7.46057
10	3.37863	7.09127	10	−0.82094	8.06563
11	4.09212	7.52390	11	−0.26832	8.64339
12	4.81764	7.93769	12	0.30244	9.19716
13	5.55379	8.33488	13	0.88921	9.72965
14	6.29942	8.71731	14	1.49025	10.24307
15	7.05355	9.08649	15	2.10414	10.73930
16	7.81540	9.44370	16	2.72968	11.21991
17	8.58427	9.79002	17	3.36586	11.68626
18	9.35957	10.12639	18	4.01180	12.13952
19	10.14080	10.45361	19	4.66674	12.58071
20	10.92751	10.77239	20	5.33003	13.01074
21	11.71931	11.08334	21	6.00107	13.43039
22	12.51585	11.38702	22	6.67935	13.84036
23	13.31682	11.68390	23	7.36442	14.24129
24	12.12194	11.97442	24	8.05585	14.63374
25	14.93097	12.25897	25	8.75327	15.01822
30	19.02744	13.60343	30	12.31949	16.83573
35	23.19492	14.84000	35	15.99563	18.50828
40	27.41892	15.99107	40	19.75974	20.06558
45	31.68930	17.07221	45	23.59633	21.52848
50	35.99862	18.09478	50	27.49400	22.91220

TABLE M2 (*cont.*)

$\mathfrak{s}_3(\lambda)$			$\mathfrak{s}_4(\lambda)$		
λ	σ	t	λ	σ	t
2	−5.90041	1.19521	2	−7.71370	0.47288
3	−6.03282	2.39330	3	−8.04938	1.78032
4	−5.98035	3.45439	4	−8.16969	2.93895
5	−5.81495	4.41961	5	−8.15190	3.99890
6	−5.57166	5.31319	6	−8.03936	4.98352
7	−5.27047	6.14692	7	−7.85680	5.90778
8	−4.92399	6.93397	8	−7.61978	6.78206
9	−4.54079	7.68074	9	−7.33885	7.61394
10	−4.12697	8.39283	10	−7.02150	8.40915
11	−3.68706	9.07462	11	−6.67328	9.17221
12	−3.22455	9.72963	12	−6.29844	9.90672
13	−2.74218	10.36072	13	−5.90030	10.61566
14	−2.24214	10.97027	14	−5.48153	11.30149
15	−1.72625	11.56030	15	−5.04430	11.96630
16	−1.19602	12.13251	16	−4.59042	12.61187
17	−0.65271	12.68839	17	−4.12141	13.23974
18	−0.09741	13.22922	18	−3.63854	13.85125
19	0.46894	13.75612	19	−3.14293	14.44758
20	1.04552	14.27010	20	−2.63553	15.02977
21	1.63163	14.77202	21	−2.11719	15.59876
22	2.22662	15.26267	22	−1.58864	16.15537
23	2.82993	15.74276	23	−1.05054	16.70036
24	3.44107	16.21293	24	−0.50348	17.23439
25	4.05957	16.67374	25	0.05203	17.75808
30	7.24939	18.85435	26	0.61550	18.27199
35	10.57482	20.86339	27	1.18651	18.77664
40	14.00916	22.73534	28	1.76467	19.27248
45	17.53358	24.49459	29	2.34962	19.75995
50	21.13411	26.15909	30	2.94103	20.23945
			35	5.98537	22.52930
			40	9.15462	24.66516
			45	12.42733	26.67388
			50	15.78767	28.57535

TABLE M2 (*cont.*)

$\mathfrak{F}_5(\lambda)$		
λ	σ	t
3	−9.91210	1.10407
4	−10.18817	2.34009
5	−10.30294	3.47597
6	−10.30721	4.53477
7	−10.23006	5.53153
8	−10.08983	6.47674
9	−9.89889	7.37806
10	−9.66605	8.24133
11	−9.39781	9.07115
12	−9.09914	9.87121
13	−8.77394	10.64453
14	−8.42531	11.39365
15	−8.05577	12.12068
16	−7.66743	12.82746
17	−7.26204	13.51558
18	−6.84108	14.18640
19	−6.40583	14.84115
20	−5.95739	15.48090
21	−5.49672	16.10661
22	−5.02466	16.71913
23	−4.54196	17.31925
24	−4.04928	17.90766
25	−3.54722	18.48499
26	−3.03631	19.05184
27	−2.51703	19.60873
28	−1.98983	20.15614
29	−1.45510	20.69454
30	−0.91320	21.22434
35	1.89287	23.75684
40	4.83715	26.12205
45	7.89594	28.34849
50	11.05178	30.45745

interest to be able to answer the—probably difficult—question: which of the trajectories $\bar{s}_m(\lambda)$ end in a zero of $\zeta(s)$ and which do not?

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Note added in proof. After having finished the paper, the author found that Franklin [10] has calculated the zeros $\bar{s}_1(1)$ and $\bar{s}_2(1)$ of $P(s, 1)$ to seven decimals. The first five decimals agree with those given in Table M2. Also, Gronwall [11] has proved that $\bar{s}_1(1)$, $\bar{s}_2(1)$ and their complex conjugates are the only complex zeros of $P(s, 1)$.

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